ANALYSIS OF THE TRUNCATED HILBERT TRANSFORM ARISING IN LIMITED DATA TOMOGRAPHY

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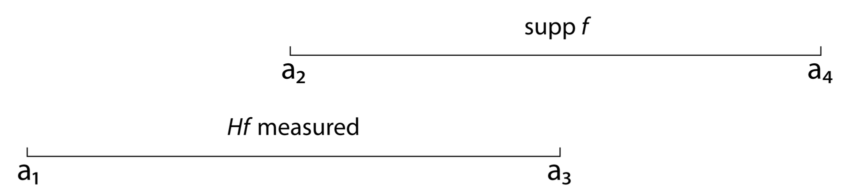
RESULTS

The truncated Hilbert transform $H_T = \mathcal{P}_{[a_1,a_3]} H \mathcal{P}_{[a_2,a_4]}$,
where $\mathcal{P}_\Omega$ is the projection onto Ω and $a_1 < a_2 < a_3 < a_4$



has the following properties:

- $H_T^* H_T$ has only discrete spectrum
- H_T is non-compact
- The problem is severely ill-posed (i.e. exponentially decaying singular values).

INTRODUCTION

Computerized Tomography

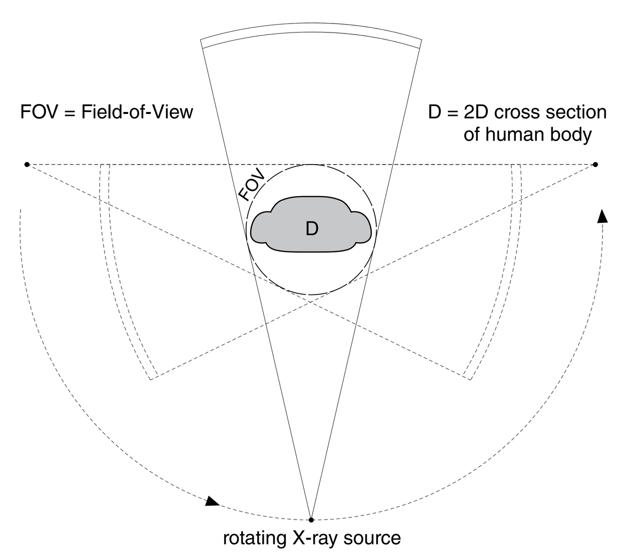


Figure: Principle of CT scanners: Rotating source allows for multiple views

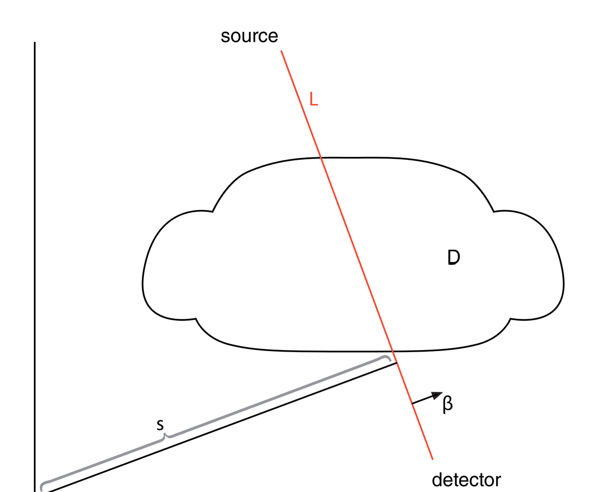


Figure: Radon measurements Detectors measure attenuation of X-ray signals

Measurements: Radon transform data

$$p(\phi, s) = \int_{-\infty}^{\infty} f(r\alpha + s\beta)dr \qquad \alpha = (\cos\phi, \sin\phi), \quad \beta = (-\sin\phi, \cos\phi)$$

$\in (0,\pi)$ $\in(-\infty,\infty)$ object density

Reconstruction: Filtered Back-Projection

Limited data scenarios

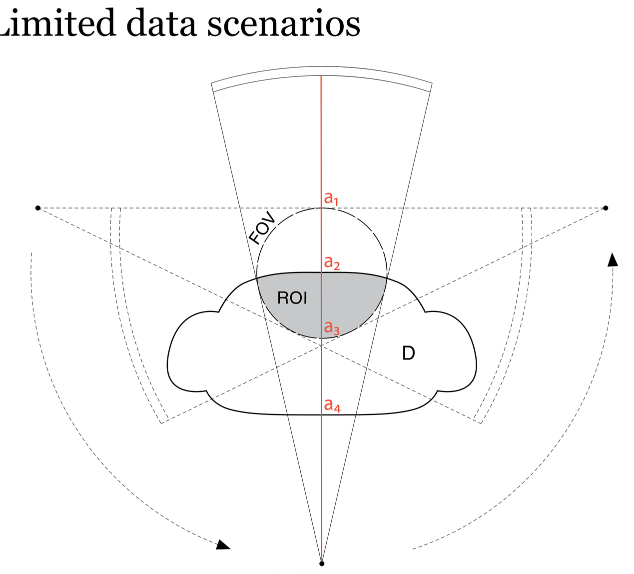


Figure: The Field-of-View (FOV) does not cover the object support D. The aim is to reconstruct within the Region-of-Interest (ROI).

See accompanying paper A1 **Reconstruction**: Differentiated Back-Projection within Region-of-Interest

2D problem $\xrightarrow{\text{reduces to family of}}$ 1D problems

Inversion of $H_T = \mathcal{P}_{[a_1,a_3]} H \mathcal{P}_{[a_2,a_4]}$ is ill-posed $\longrightarrow$ Spectral analysis

SPECTRAL ANALYSIS

See accompanying paper A2

Ingredient 1

Finding a differential operator L_S, self-adjoint, that commutes with H_T:

$$L_S H_T f = H_T L_S f, \ f \in \mathcal{D}(L_S)$$

L_S has discrete spectrum (resolvent $(L_S - i)^{-1}$ is compact)

Eigenfunctions of $L_S, \tilde{L}_S$ — Singular functions of H_T

$$\boxed{\begin{array}{l} L_S f_n = \lambda_n f_n \\ \tilde{L}_S g_n = \lambda_n g_n \end{array}} \longleftrightarrow \boxed{\begin{array}{l} H_T f_n = \sigma_n g_n \\ H_T^* g_n = \sigma_n f_n \end{array}} \qquad \begin{array}{l} f_n \in \mathcal{L}^2(a_2, a_4) \\ g_n \in \mathcal{L}^2(a_1, a_3) \end{array}$$

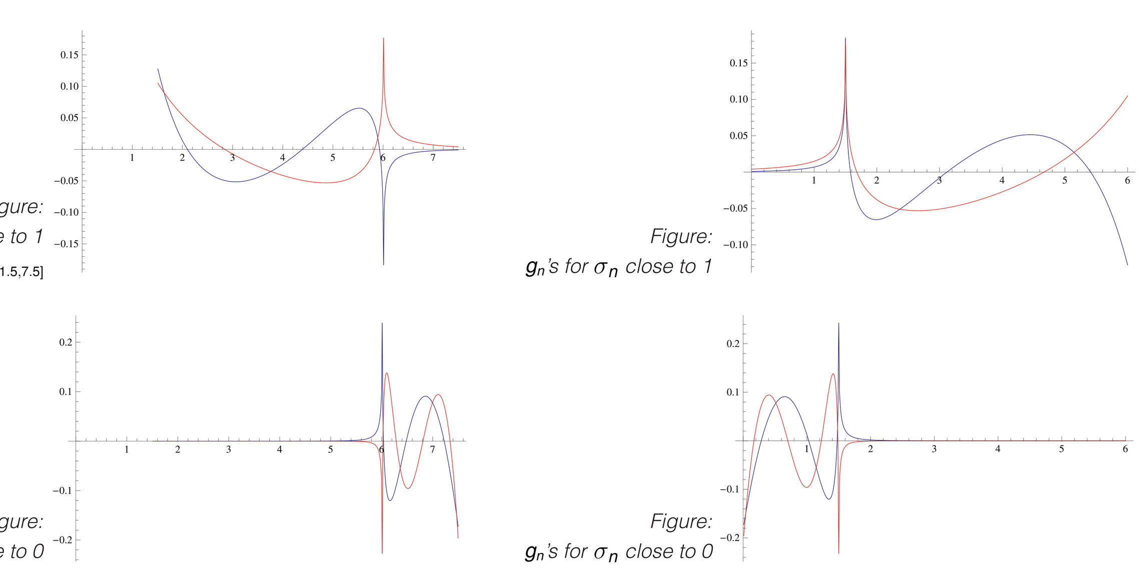


Figure: f_n's for σ_n close to 1 and $H_T = \mathcal{P}_{[0,6]} H \mathcal{P}_{[1.5,7.5]}$

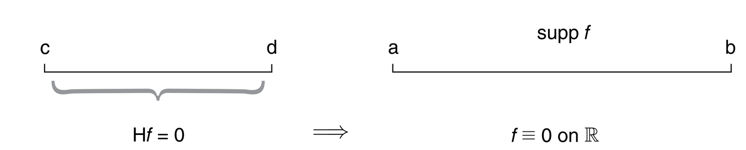
Figure: g_n's for σ_n close to 1

Figure: f_n's for σ_n close to 0

Figure: g_n's for σ_n close to 0

Ingredient 2

If $f \in \mathcal{L}^2(a,b)$, a, b finite
and $(Hf)(x) = 0$ on (c,d) open,
and (c,d) disjoint from (a,b),
then $f \equiv 0$ on $\mathbb{R}$.



Ker $H_T = \{0\}$ — Trivial nullspace → UNIQUENESS

Ran $H_T \neq \mathcal{L}^2(a_1, a_3)$ — Dense range, not all of $\mathcal{L}^2(a_1,a_3)$ → ILL-POSEDNESS

$\overline{\text{Ran } H_T} = \mathcal{L}^2(a_1, a_3)$

Ingredient 3

$\|H_T^* H_T\| = 1$

The singular values of H_T accumulate only at 0 and 1.
0 and 1 are not singular values.

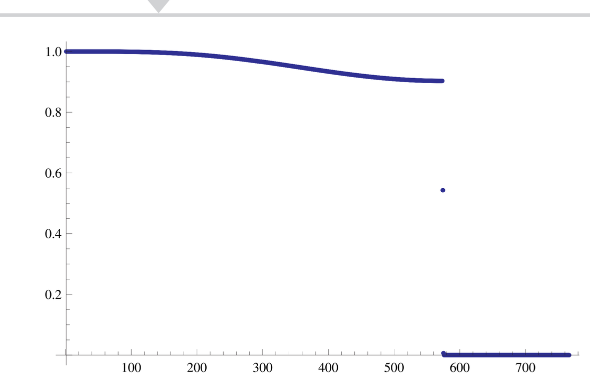


Figure: Distribution of singular values for a discretization of $H_T^* H_T$.

Question

How fast is convergence to 0 and 1? $\xrightarrow{\text{requires}}$ Asymptotic analysis

ASYMPTOTIC ANALYSIS

Ingredient 1

Spectrum of L_S accumulates at $+\infty$ and $-\infty$.

Ingredient 2

$L_S f_n = \lambda_n f_n \qquad \tilde{L}_S g_n = \lambda_n g_n$

Asymptotic behavior of f_n, g_n, for $|\lambda_n|$ large:

- away from a_i: WKB approximation
- close to a_i: Bessel approximation

Ingredient 3

See accompanying paper A3

Correct matching of these approximations requires:

- Uniqueness of Riemann-Hilbert problem
- Plemelj-Sokhotski formula

to ensure boundary and transmission conditions at singular points a_i.

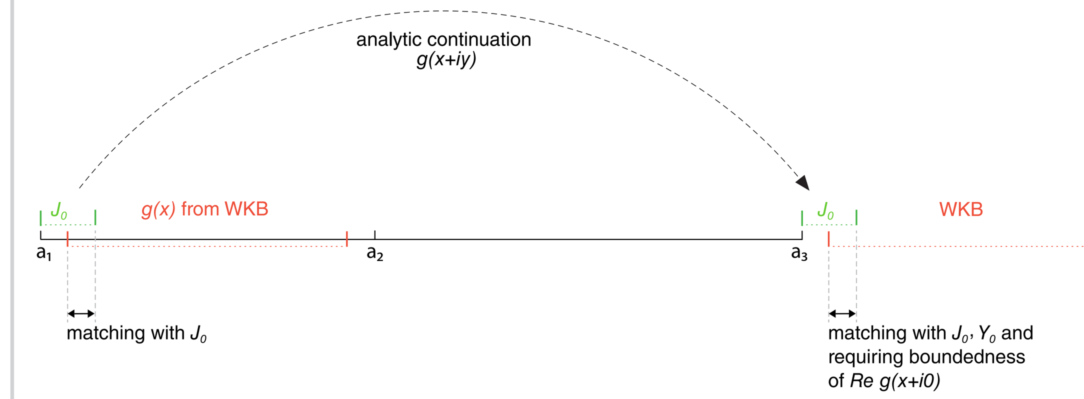


Figure: Construction of singular functions from asymptotics of Sturm-Liouville eigenfunctions.

Asymptotic behavior of eigenfunctions

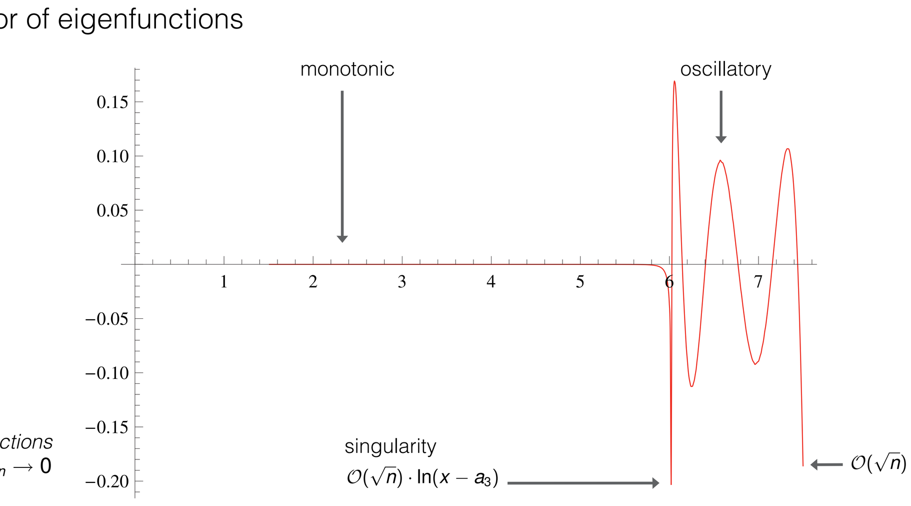


Figure: Eigenfunctions f_n for $\sigma_n \to 0$

Ingredient 4

Estimate logarithmic terms in $(Hg_n)(a_3^+)$ and $f_n(a_3^+)$ and find asymptotic behavior of $(Hg_n)(a_3^+)/f_n(a_3^+)$.

Asymptotic behavior of singular values $\sigma_n \to 0$:

$$\sigma_n = 2e^{-cn\pi} \cdot \left(1 + \mathcal{O}(n^{\delta - 1/2})\right)$$

$c = c(a_1, a_2, a_3, a_4) > 0$, $0 < \delta \ll 1$

The problem is severely ill-posed.

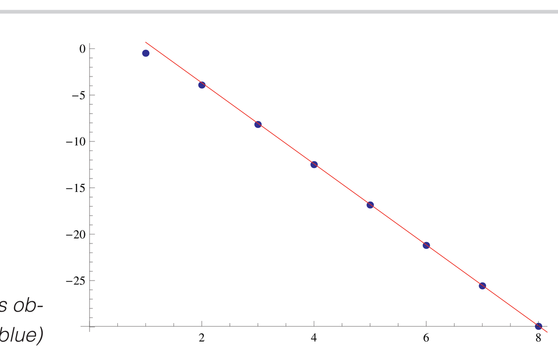


Figure: Logarithmic plot of singular values obtained from theory (red) and numerics (blue)

CONTACT INFO

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References
[1] R Al-Aifari, A Katsevich, Spectral analysis of the truncated Hilbert transform with overlap, 2013, submitted. Preprint: arXiv: 1302.6295
[2] A Katesevich, A Tovbis, Finite Hilbert transform with incomplete data: nullspace and singular values, 2012, Inverse Problems 28 105006 (28pp.)

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